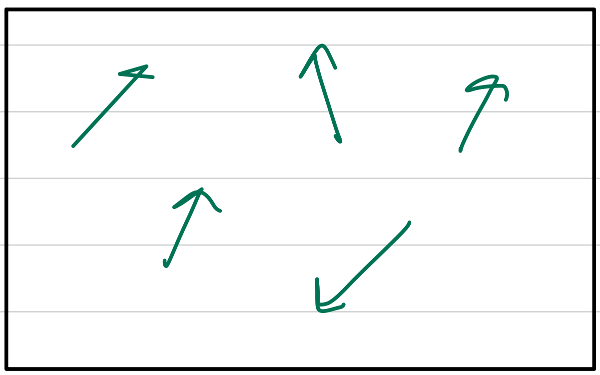
Degenerate Fermi Gas

Lets look at a classical gas and understand why the approach does not apply for electrons in a metal



For a classical approach to work the interparticle spacing

$$l_o \equiv \left(\frac{N}{V}\right)^{1/3}$$

Should be large compared to a thermal debroglie wavelength $\lambda \equiv \frac{\hbar}{(2\pi m_e kT)^{1/2}}$. In tems of the density we have

$$\boxed{n \ll n_Q}$$

$n \equiv N/V$

The quantum concentration $n_Q \equiv \frac{1}{\lambda^3} = \frac{(2\pi m kT)^{3/2}}{h^3}$

i.e. the density should be low enough that the de Broglie waves don't overlapp.

This can also be expressed in terms of the chemical potential. We showed for a classical gas, that

$$e^{\beta\mu} = \frac{n}{n_Q} \qquad \text{or} \qquad \mu = kT \ln \frac{n}{n_Q}$$

So the classical limit is when

$$\boxed{e^{\beta\mu} \ll 1}$$

Classical limit

Lets check out the Bose and Fermi distributions in the classical limit:

$$n_{FD}(\varepsilon) = \frac{1}{e^{\beta(\varepsilon-\mu)}+1} = \frac{\overbrace{e^{\beta\mu}e^{-\beta\varepsilon}}^{\text{small}}}{1+\underbrace{e^{\beta\mu}e^{-\beta\varepsilon}}_{\text{small}}}$$

$$\simeq e^{\beta\mu}e^{-\beta\varepsilon} \ll 1$$

The Bosons work similarly

$$n_{BE}(\varepsilon) = \frac{1}{e^{\beta(\varepsilon-\mu)}-1} = \frac{\overbrace{e^{\beta\mu}e^{-\beta\varepsilon}}^{\text{small}}}{1-\underbrace{e^{\beta\mu}e^{-\beta\varepsilon}}_{\text{small}}}$$

$$\simeq e^{\beta\mu}e^{-\beta\varepsilon} \ll 1$$

So in the classical limit the modes are mostly unoccupied. The distribution function in this limit is known as the Boltzmann distribution

$$\boxed{n_{Boltz}(\varepsilon) = e^{\beta\mu}\,e^{-\beta\varepsilon}}$$

We will see that this reproduces all results of a classical gas.

Now look at Cu, wich has a density of $\rho \simeq 9\, g/cm^3$, with one valence electron per nucleus. So the density of valence electrons is

$$n = \frac{N}{V} = \frac{9\, g/cm^3}{64\, g} \times 1\,mol = 8.4 \times 10^{22} \frac{1}{cm^3} = 84 \frac{1}{nm^3}$$

while

$$n_Q = \frac{(2\pi m_e kT)^{3/2}}{h^3} = \frac{(2\pi m_e c^2 kT)^{3/2}}{(hc)^3} = 0.01 \frac{1}{nm^3}$$

So, $n \gg n_Q$, and the electron gas is very far from a classical one! The relevant physics in this regime is when

$$\boxed{e^{\beta\mu} \gg 1 \quad \text{or} \quad \frac{\mu}{kT} \gg 1}$$

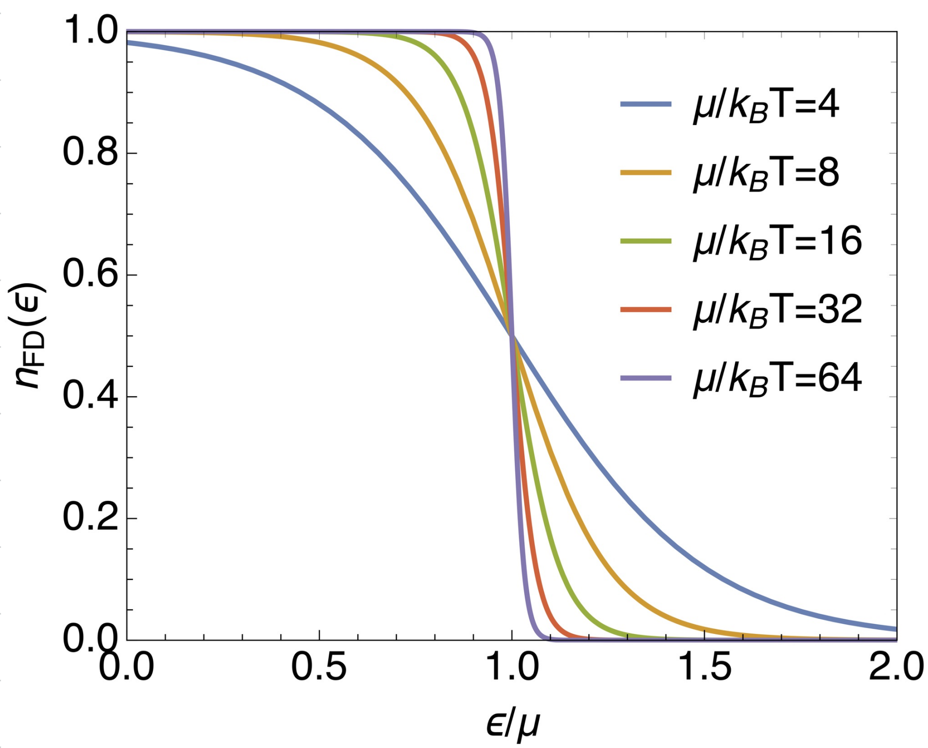
degenerate quantum regime

this is a low temperature limit.

The Fermi Distribution

The figure below shows the Fermi-Dirac distribution for large values of μ/kT

$$n_{FD} = \frac{1}{e^{\beta(\varepsilon-\mu)}+1}$$



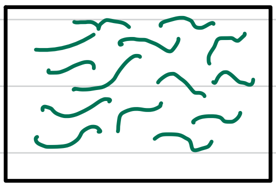
We see that at large μ/kT the fermi distribution becomes a step function

$$\boxed{n_{FD}(\varepsilon) = \begin{cases} 1 & \text{if} \quad \varepsilon < \mu_0 \\ 0 & \text{if} \quad \varepsilon > \mu_0 \end{cases}}$$

This is the $T=0$ limit. I put μ_0 instead of just μ to remind this.

Then all modes with energy less than μ are filled

A Box of Electrons



N electrons per volume V, what is the energy U and pressure. As we will see these are both non-zero even at zero T

$\varepsilon = p^2/2m$ for non-relativistic particles.

$$N = \sum_{\text{modes}} n_{FD}(\varepsilon)$$

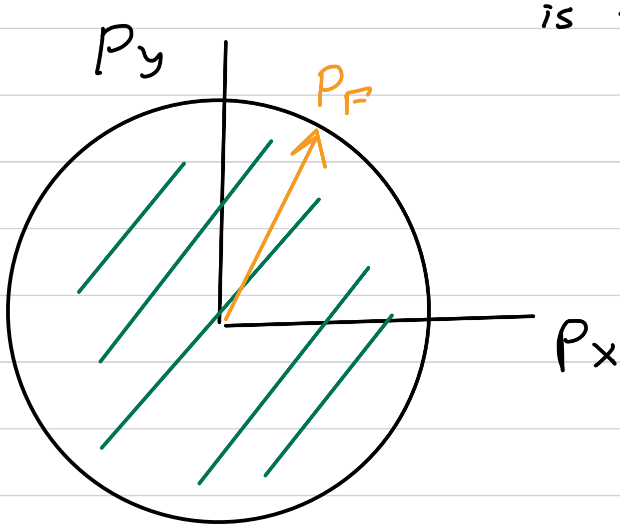
filled mode

$$= g\, V \int \frac{d^3p}{(2\pi\hbar)^3} \cdot \begin{cases} 1 & \text{if } p^2/2m < \mu_0 \\ 0 & \text{if } p^2/2m > \mu_0 \end{cases}$$

empty mode

spin degeneracy $g=2$

So the momentum ranges up to a maximum momentum $p_F \equiv (2m\mu_0)^{1/2}$, known as the "Fermi-Momentum", that is set by the chemical potential. The maximum energy of is the "Fermi-Energy", $\varepsilon_F = \dfrac{p_F^2}{2m} = \mu_0$



$$p_F = \sqrt{2m\mu_0}$$

$$\varepsilon_F = \frac{p_F^2}{2m} = \mu_0$$

The integral over d^3p is just the volume of the sphere

$$N = \frac{gV}{(2\pi\hbar)^3}\,\frac{4}{3}\pi p_F^3$$

this gives the relation between μ and the density in the degenerate case.

So

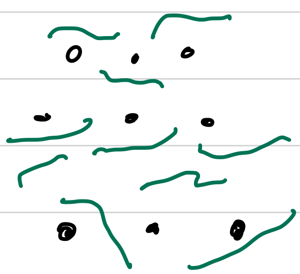
We will interpret this shortly!

$$\frac{N}{V} = \frac{g}{6\pi^2}\left(\frac{p_F}{\hbar}\right)^3 \quad \text{or} \quad p_F = \hbar\left(\frac{6\pi^2}{g}\frac{N}{V}\right)^{1/3}$$

The fermi-energy or chemical potential

$$\boxed{\mu_0 = \varepsilon_F = \frac{p_F^2}{2m} = \frac{\hbar^2}{2m}\left(\frac{6\pi^2}{g}\frac{N}{V}\right)^{2/3}} \qquad \mu_0 \propto \left(\frac{N}{V}\right)^{2/3}$$

Picture



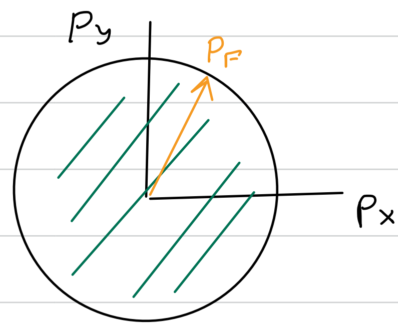
As the density increases the wavelength becomes of order the interparticle spacing

$$\lambda_F \equiv \frac{\hbar}{p_F} = \left(\frac{gV}{6\pi^2 N}\right)^{1/3} \sim \left(\frac{V}{N}\right)^{1/3}$$

The typical momentum then increases with the density, i.e the density increases, the wavelength gets shorter and by quantum mechanics the momentum goes up.

$$p_F \sim \hbar\left(\frac{N}{V}\right)^{1/3}$$

$$\varepsilon_F \sim \frac{\hbar^2}{2m}\left(\frac{N}{V}\right)^{2/3}$$



All momentum states below p_F are filled.

The energy of the electrons follows similarly

$$U = \sum_{\text{modes}} n_{FD}(\varepsilon)\,\varepsilon$$

$$= gV \int \frac{d^3p}{(2\pi\hbar)^3} \cdot \begin{cases} 1 & \text{if } p^2/2m < \mu_0 \\ 0 & \text{otherwise} \end{cases} \cdot \frac{p^2}{2m}$$

Writing $d^3p \to 4\pi p^2\,dp$

$$U = \frac{gV}{2\pi^2} \frac{1}{\hbar^3} \int_0^{p_F} \frac{p^4\,dp}{2m}$$

$$= \frac{gV}{2\pi^2} \frac{1}{5} \left(\frac{p_F}{\hbar}\right)^3 \frac{p_F^2}{2m}$$

now $N = \frac{gV}{2\pi^2} \frac{1}{3} \left(\frac{p_F}{\hbar}\right)^3$

$$\boxed{U = \frac{3}{5} N \varepsilon_F}$$

Note $\varepsilon_F \propto (N/V)^{2/3}$ so the energy per volume U/V scales as

$$\frac{U}{V} \propto \left(\frac{N}{V}\right)^{5/3}$$

independent of temperature. Contrast this with an ideal gas where $\frac{U}{V} = \frac{3}{2}\frac{N}{V} kT$

Copper Example:

Lets calculate the fermi energy of Cu and compare ε_F to $k_B T$. The electron density N/V can be deduced from the mass density $\rho = 8.96\ g/cm^3$, and the assumption that there is one valence electron per nuclei. It is the density of these electrons that we are calculating. Cu has a molar mass of 64 g

$$\frac{N}{V} = \frac{8.96}{64g} \cdot \frac{6.02 \times 10^{23}}{cm^3} = 8.4 \times 10^{22} \frac{1}{cm^3}$$

$$\varepsilon_F = \frac{\hbar^2}{2m}\left(\frac{6\pi^2}{2} \cdot \frac{N}{V}\right)^{2/3}$$

spin

To estimate the energy we insert the Bohr Radius $a_0 = 0.52\ \text{Å}$ and recognize the Rydberg constant

$$\frac{\hbar^2}{2ma_0^2} = 13.6 eV$$

So

$$\varepsilon_F = \frac{\hbar^2}{2ma_0^2}\left(3\pi^2 \frac{Na_0^3}{V}\right)^{2/3} = 13.6eV \cdot 0.51 \simeq 7.0eV$$

$$\frac{\varepsilon_F}{k_B T} \simeq \frac{7.0eV}{1/40\ eV} \simeq 280$$

this is a large number And approximating $n_{FD}(\varepsilon)$ by a step-function is good.

Thermodynamics of A degenerate gas

Now

work at zero temperature

$$dU = T\,\cancel{dS} - p\,dV + \mu\,dN \qquad (dS \to 0)$$

So we can find the pressure of the gas by differentiation

$$p = -\left(\frac{\partial U}{\partial V}\right)_N \quad \text{and} \quad \mu = \left(\frac{\partial U}{\partial N}\right)_V$$

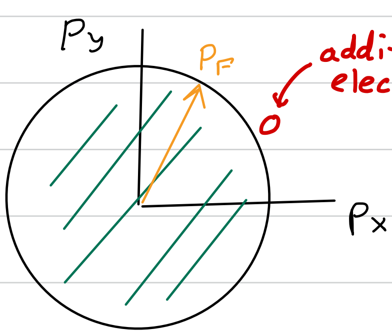
- We have $U = c_0 N \left(\frac{N}{V}\right)^{2/3}$ where $c_0 = \text{const}$. Differentiating

$$p = -\frac{\partial U}{\partial V} = +\frac{2}{3} c_0 N\, N^{2/3}\, V^{-2/3-1},$$ and so:

compare $p = \frac{N}{V} kT$

$$\boxed{p = \frac{2}{3}\frac{U}{V} \quad \text{or} \quad p = \frac{2}{5}\frac{N}{V}\varepsilon_F}$$

- For the chemical potential we expect to find $\varepsilon_F = \mu$.



Since if I add one additional electron, it will go to the fermi surface, increasing the energy of the system by ε_F

$$\Delta U = \mu\,\Delta N$$ ← $\Delta N = 1$ here

Since $U = c_0 \frac{N^{5/3}}{V^{2/3}}$, which is differentiated, $\mu = \frac{\partial U}{\partial N}$:

$$\mu = \frac{5}{3} c_0 \frac{N^{5/3}}{V^{2/3}} \frac{1}{N} = \frac{5}{3} \frac{U}{N} = \frac{\cancel{5}}{\cancel{3}} \cdot \frac{\cancel{3}}{\cancel{5}} \varepsilon_F = \varepsilon_F$$

Which confirms our intuition.

Discussion

The degenerate fermi gas is so important and is responsible for the pressure in neutron stars and white dwarfs. It is also technically important for many electronic instruments based on GaAs.