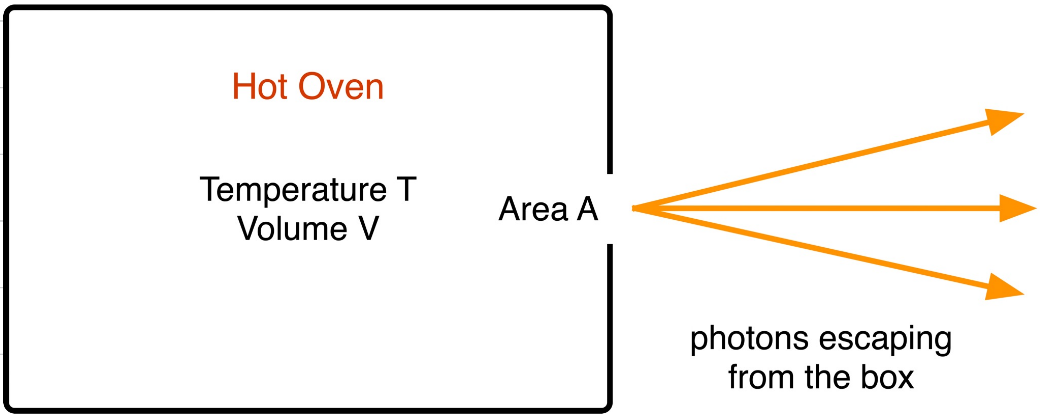
Heat From an Oven

How many photons or energy leave the box per area per second ?



Given a hot oven we want to compute the energy emitted per area per second, Φ_E. Historically, theorists tried to compute Φ_E and found ∞, because the concept of photons $E_\gamma = hf$ had not been invented. It was found experimentally that

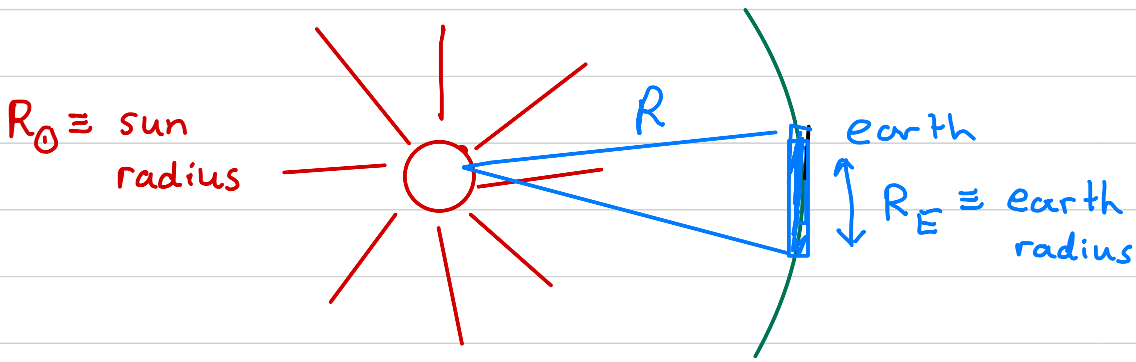
$$\boxed{\Phi_E = \sigma T^4 \qquad \text{with} \qquad \sigma = 5.67 \times 10^{-8} \frac{W}{m^2 (^\circ K)^4}}$$

This is the blackbody radiation law.

We will compute σ in a few pages, but first let's work a practical example.

Example

The energy per area per second absorbed by the earth from the sun is $1 kW/m^2$. Estimate the temperature of the sun.



The sun emits a total energy per second, dU/dt, of

$$\frac{dU}{dt} = \sigma T^4 \, 4\pi R_\odot^2$$

This gets spread out over a sphere of radius R. So the energy per area on that sphere (which includes the earth) is

$$I = \frac{1}{A}\frac{dU}{dt} = \sigma T^4 \frac{4\pi R_\odot^2}{4\pi R^2} = \frac{1\,kW}{m^2}$$

So solving for T

$$T = \left(\frac{I}{\sigma}\frac{R^2}{R_\odot^2}\right)^{1/4} = \left(\frac{I}{\sigma}\pi\frac{R^2}{\pi R_\odot^2}\right)^{1/4} \equiv \left(\frac{I\,\pi}{\sigma\,\Omega}\right)^{1/4}$$

Here we have defined the "solid-angle" of the sun

$$\Omega \equiv \frac{A}{R^2} \equiv \frac{\text{Area of patch on sphere}}{(\text{radius})^2}$$

This can be measured with a protactor: $\Omega/4\pi$ is the fraction of the sky covered by the sun. $\Omega = 6.8 \times 10^{-5}$. Substituting Ω, and σ, we find

$$\boxed{T = 5340°K}$$

pretty close!

Flux of Photons: The computation

Given a box of photons (an oven), we want to compute the number of photons which are emitted per unit time per unit area. The photons are "flying" with a range of angles θ and ϕ

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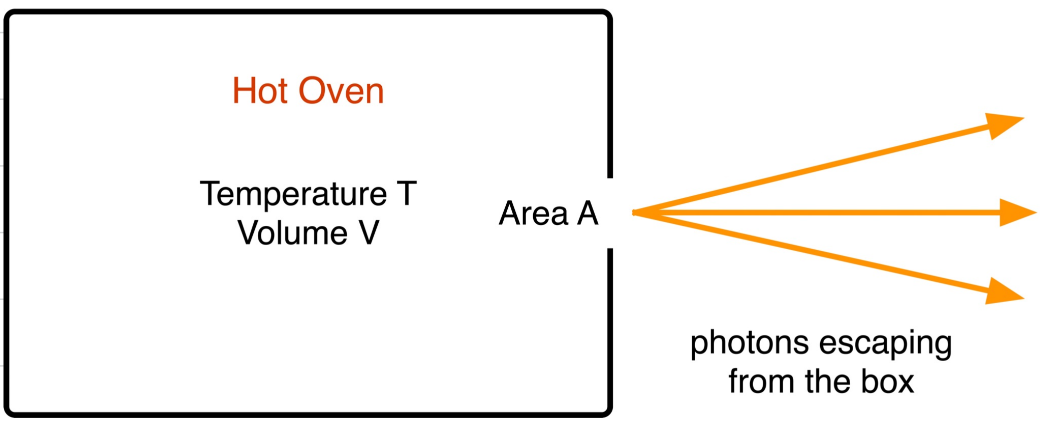


The number of photons is

$$N = 2V \int \frac{d^3 p}{(2\pi\hbar)^3} \frac{1}{e^{cp/kT} - 1}$$

So the number of photons per volume n_γ with momentum $\vec{p} = (p_x, p_y, p_z)$ in range $[p_x, dp_x]$, $[p_y, dp_y]$, $[p_z, dp_z]$ is

$$dn_\gamma = \frac{2}{e^{cp/kT} - 1} \frac{dp_x \, dp_y \, dp_z}{(2\pi\hbar)^3}$$

So the number of photons with momentum magnitude in $[p, dp]$ flying in angular range is $[\theta, d\theta]$, $[\phi, d\phi]$

spherical coordinates

$$dn_\gamma = \frac{2}{e^{cp/kT} - 1} \frac{p^2\,dp\,\sin\theta\,d\theta\,d\phi}{(2\pi\hbar)^3}$$

uniform over solid angle

$$= \frac{2}{(2\pi\hbar)^3} \frac{(4\pi p^2\,dp)}{e^{cp/kT} - 1} \times \frac{\sin\theta\,d\theta\,d\phi}{4\pi}$$

If we don't care about momentum magnitude we may integrate over p, yielding

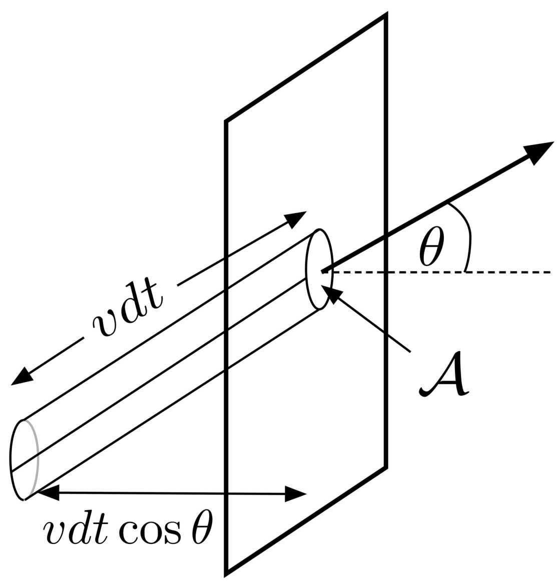
same density we found previously, times a distribution uniformly distributed over the solid angle

$$dn_\gamma = n_\gamma \frac{\sin\theta\,d\theta\,d\phi}{4\pi}$$

$$n_\gamma = 2\int_0^\infty \frac{1}{(e^{cp/kT}-1)} \frac{4\pi p^2\,dp}{(2\pi\hbar)^3}$$

Flux

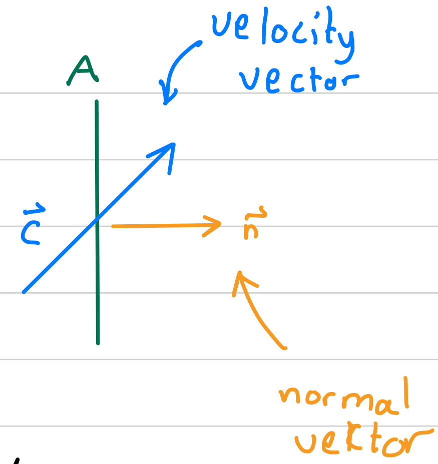
Now take a time interval Δt and an angular range $[\theta, d\theta]$ and $[\phi, d\phi]$. Then look at the picture below. In dt all of the photons in the tube of length $c\Delta t$ will cross through the hole



$v = c$

$=$ speed of particles.

The tube volume is



$$\text{Volume} = \text{base} \times \text{height}$$

$$= A \cdot c\, dt \cos\theta$$

And so the number of photons crossing the hole is

$$\Delta N_\gamma = dn_\gamma \cdot A\, c \cos\theta\, dt = dn_\gamma\, \vec{c} \cdot \vec{n}\, A\, dt$$

see picture

And the flux of particles (the number per area per second with angular range in $[\theta, d\theta]$ and $[\phi, d\phi]$) is

$$\boxed{d\Phi_N = \frac{dN_\gamma}{A\, dt} = dn_\gamma\, c\cos\theta = dn_\gamma\, \vec{c} \cdot \vec{n}}$$

So now we need to integrate over angles θ and ϕ

$$\Phi_N = \int_0^{\pi/2} d\theta \int_0^{2\pi} d\phi\; n_\gamma \frac{\sin\theta\, d\theta\, d\phi}{4\pi} \cdot c\cos\theta$$

$$\boxed{\Phi_N = \frac{1}{4} n_\gamma c}$$

this is the total number of photons escaping from the hot oven per area per second

Energy Flux

Usually we worry more about the energy carried away by the escaping photons. A photon with momentum p carries energy cp. The energy per volume with momentum between p and p+dp and an angular range [θ, dθ] and [φ, dφ] is

$$du_\gamma = \frac{cp}{e^{cp/kT}-1} \cdot \frac{2d^3p}{(2\pi\hbar)^3}$$

Writing $d^3p = 4\pi p^2 dp \dfrac{d\Omega}{4\pi}$ with $d\Omega = \sin\theta \, d\theta \, d\phi$ and integrating over momenta yields

this is the energy per volume we found previously

$$dU_\gamma = u_\gamma \frac{\sin\theta \, d\theta \, d\phi}{4\pi}$$

$$u_\gamma = \frac{\pi^2}{15}\left(\frac{kT}{\hbar c}\right)^3 kT$$

Now the computation goes as before the energy crossing the hole in time dt and area A is dU_γ

$$dU_\gamma = du_\gamma \cdot c \, dt \cos\theta \, A$$

And integrating over angles like before gives

$$\Phi_E = \frac{1}{A}\frac{dU_\gamma}{dt} = \frac{1}{4} u_\gamma c$$

So since

$$u_\gamma = \frac{\pi^2}{15}\left(\frac{kT}{\hbar c}\right)^3 kT$$

we get

$$\boxed{\begin{gathered} \text{Energy flux} = \frac{1}{A}\frac{\Delta u}{\Delta t} = \sigma T^4 \\ \sigma = \left(\frac{k_B}{\hbar c}\right)^3 k_B c \frac{\pi^2}{60} = 5.67\times10^{-8}\ \frac{W}{m^2\ {}^\circ K^4} \end{gathered}}$$