The Pressure and Free Energy

- The grand partition function for one bosonic mode is

$$\boxed{Z_{G1} = \frac{1}{1 - e^{-\beta(\varepsilon_1 - \mu)}}}$$

this is the energy of the first mode

- And the corresponding grand potential is $\Phi_1 = -kT \ln Z_{G1}$
We showed at the beginning that if I have a collection of modes $\ell = 1, 2, 3$ for example the $Z_G = Z_{G1} Z_{G2} Z_{G3}$. Thus the grand potential, $\Phi_G = -kT \ln Z_G$ is a sum

$$\Phi_G = \sum_{\text{modes}} \Phi_{G\ell}$$

grand potential for ℓ-th mode

$$= \sum_{\text{modes}} kT \ln\left(1 - e^{-\beta(\varepsilon(p) - \mu)}\right)$$

- Once we know the grand potential we can find the pressure entropy and number by differentiation, $\Phi_G = U - TS - \mu N$:

$$d\Phi_G = -S\,dT - p\,dV - N\,d\mu$$

use me in Homework.

so $p = -(\partial \Phi_G / \partial V)_{T,\mu}$. We will use this now

- For photons $\varepsilon(p) = cp$, the chemical potential $\mu = 0$, and

$$\sum_{\text{modes}} = 2 \int \frac{V\,d^3p}{(2\pi\hbar)^3}$$

• So

$$\Phi_G = 2V \int \frac{d^3p}{(2\pi\hbar)^3}\, kT \ln\left(1 - e^{-\beta c p}\right)$$

$$= \frac{V}{\pi^2\hbar^3} \int_0^\infty p^2\,dp\; kT \ln\left(1 - e^{-\beta c p}\right)$$

Use

$d^3p \Rightarrow 4\pi p^2\,dp$

you could make it dimensionless & do it numerically

Now integrate by parts: $\int u\,dv = uv - \int v\,du$

$dv = p^2 dp \qquad u = kT\ln(1 - e^{-\beta c p})$

$v = \frac{1}{3}p^3 \qquad du = \frac{kT\, e^{-\beta c p}\,\beta c\,dp}{(1 - e^{-\beta c p})} = \frac{c\,dp}{e^{\beta c p} - 1}$

you have to do this yourself.

So

$$\Phi_G = \frac{V}{\pi^2\hbar^3}\left[\frac{1}{3}p^3\, kT\ln(1 - e^{-\beta c p})\Big|_0^\infty - \int_0^\infty \frac{c\,dp}{e^{\beta c p} - 1}\cdot\frac{1}{3}p^3 \right]$$

the bndry term vanishes for $p = 0$ and $p \to \infty$

$$= \frac{-V}{\pi^2\hbar^3}\left[\frac{1}{3}\int_0^\infty \frac{c\,dp}{e^{cp/kT} - 1}\, p^3\right]$$

$$= -V\left(\frac{kT}{\hbar c}\right)^3 kT \left(\frac{\pi^2}{45}\right)$$

Use $\int_0^\infty dx\, x^3/e^x - 1 = \frac{\pi^4}{15}$

A table of such integrals will be given.

Now we may use $p = -(\partial \Phi_G / \partial V)_T$ so

$$\boxed{P = \left(\frac{kT}{\hbar c}\right)^3 kT \, \frac{\pi^2}{15}}$$

this the pressure from a gas of photons

Finally we notice that the integral in the pressure $\int_0^\infty dx \, x^3/e^x - 1$ came up before when computing the energy density.
In fact we have

$$\boxed{P = \frac{1}{3}\left(\frac{U}{V}\right)}$$

the equation of state of an ideal relativistic gas of photons is pressure is 1/3 of the energy density.