Independent variables

- Consider random variable x, and random variable y, the probability that x is between $x+dx$ and y is between $y+dy$ is

$$d\mathcal{P} = P(x,y)\, dx\, dy$$

If x, and y are independent then the $P(x,y)$ factorizes

$$d\mathcal{P} = P_x(x)\, dx\, P_y(y)\, dy$$

Then

$$\langle xy \rangle = \int dx\, dy\; P(x,y)\; x\, y$$

$$= \int dx\, dy\; P_x(x)\, P_y(y)\, x\, y$$

$$= \int dx\, P_x(x)\, x \int dy\, P_y(y)\, dy\, y$$

$$= \langle x \rangle \langle y \rangle$$

Sums of independent variables

- Consider

$$Y = x_1 + x_2 + \dots x_n$$

where each x is drawn independently
from the distribution $P(x)$

i.e $P(x_1, \dots x_n) = P(x_1) P(x_2) \dots P(x_n)\, dx_1\, dx_2 \dots dx_n$

Then

$$\langle Y \rangle = \langle x_1 + \dots + x_n \rangle$$

Clear n times the average of $\langle x \rangle$

$$\boxed{\langle Y \rangle = \langle x_1 \rangle + \langle x_2 \rangle + \dots \langle x_n \rangle = n \langle x \rangle}$$

What about the Variance of $\langle Y \rangle$?

$$\delta Y = Y - \langle Y \rangle$$

$$= (x_1 - \langle x_1 \rangle) + (x_2 - \langle x_2 \rangle) + \dots x_n - \langle x_n \rangle$$

$$\delta Y = \delta x_1 + \delta x_2 + \dots + \delta x_n$$

$$\langle \delta Y^2 \rangle = \langle (\delta x_1 + \delta x_2 + \dots \delta x_n)^2 \rangle$$

$$= \langle \delta x_1^2 \rangle + \langle \delta x_2^2 \rangle + \dots \langle \delta x_n^2 \rangle + \text{terms like } \langle \delta x_1\, \delta x_2 \rangle$$

But the cross terms all vanish since

$$\langle \delta x_1\, \delta x_2 \rangle = \langle \delta x_1 \rangle \langle \delta x_2 \rangle = 0$$

independence

So

$$\boxed{\sigma_Y^2 = \langle \delta Y^2 \rangle = n \langle x^2 \rangle = n \sigma_x^2}$$

- There is more that can be said. I will not prove it, but if n is large the probability of Y is gaussian, regardless of $P(x)$!

This is the Central Limit Theorem

$$\boxed{\begin{gathered} P(Y) = \frac{1}{\sqrt{2\pi\sigma_Y^2}} e^{-(Y-\bar{Y})^2/2\sigma_Y^2} \\ \text{with} \quad \sigma_Y^2 = n\sigma_x^2 \qquad \bar{Y} = n\langle x \rangle \end{gathered}}$$