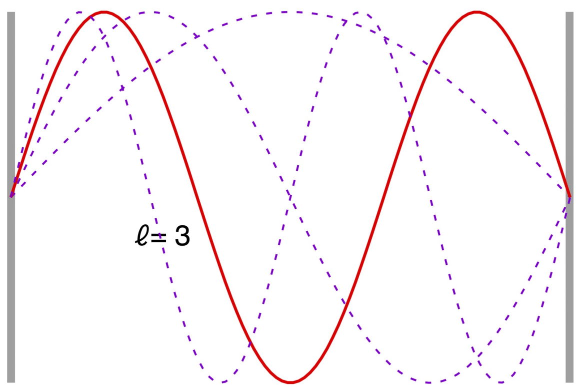
Quantum Gasses

- ## Single Particle Energy Levels

In the quantum mechanics of a single particle we determine the wave-functions, or single particle (eigen) modes, $\psi_\ell(x)$ and single particle energy levels ε_ℓ, where ℓ labels the quantum state. (We use "ℓ" here instead of n ($\psi_n(x)$, ε_n) because we will need n below). For the particle in the box the first four wave functions are shown below with the $\ell = 3$ mode emphasized. For the particle in the box, these states are labelled by their momenta p



$$\varepsilon(p) \equiv \varepsilon_\ell \equiv \frac{p_\ell^2}{2m} \quad \text{with} \quad p_\ell = \frac{\hbar \pi}{L} \ell \qquad \ell = 1, 2, 3, \ldots$$

$$= \frac{\hbar^2 \pi^2}{2mL^2} \ell^2$$

In three dimensions the box states are labeled by three quantum numbers (ℓ_x, ℓ_y, ℓ_z), which again parametrizes the momentum of the state

$$\vec{p}_\ell = \frac{\hbar \pi}{L} (\ell_x, \ell_y, \ell_z)$$

ℓ_x, ℓ_y, ℓ_z are integers

Occupation Numbers

To describe a system of many particles, we use the occupation numbers. For a system of three modes $\varepsilon_1, \varepsilon_2, \varepsilon_3$ the number of particles in each mode is labelled n_1, n_2, n_3 and these "occupation numbers" label the states, S, of the system. So if the state is $n_1=0, n_2=2, n_3=1$, we have no particles in state 1, two particles in 2, and one particle in 3, etc. The Energy and particle number of the states are E_S and N_S

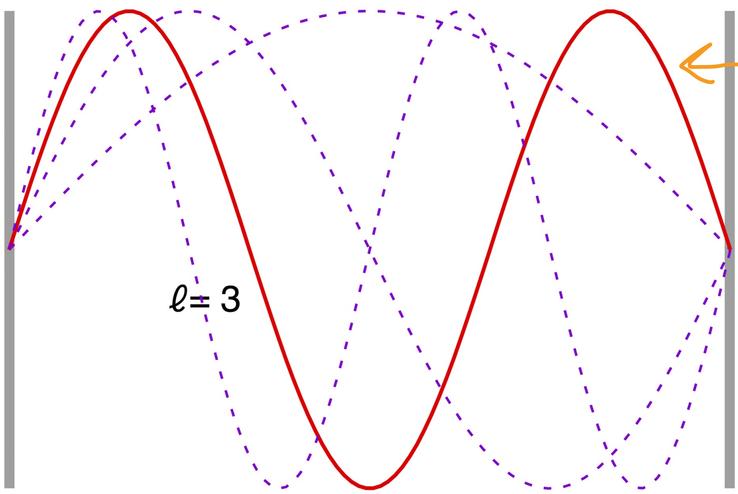
$$N_S = n_1 + n_2 + n_3$$

$$E_S = n_1\varepsilon_1 + n_2\varepsilon_2 + n_3\varepsilon_3$$

Thus the energy in mode 3 is $n_3\varepsilon_3$. The mean energy in mode 3 is $\bar{n}_3\varepsilon_3$. Only the occupation numbers change, not the modes themselves.

Statistical Mechanics

We regard each mode of the cavity as an independent statistical system, Stealing energy and particles from other modes and the environment.



this mode is sharing energy and particles from other modes. A mode is bright if it has a lot of particles (aka quanta)

The grand partition function of the system is:

$$Z_G = \sum_{\text{states}} e^{-\beta(E_s - \mu N_s)}$$

$$= \sum_{n_1, n_2, n_3} e^{-\beta(n_1\varepsilon_1 + n_2\varepsilon_2 + n_3\varepsilon_3 - \mu n_1 - \mu n_2 - \mu n_3)}$$

$$= \sum_{n_1} e^{-\beta n_1(\varepsilon_1 - \mu)} \sum_{n_2} e^{-\beta n_2(\varepsilon - \mu)} \sum_{n_3} e^{-\beta n_3(\varepsilon_3 - \mu)}$$

$$Z_G = Z_{G1} \cdot Z_{G2} \cdot Z_{G3}$$

Thus we see that the partition function of the full system factorizes into a product, one partition function for each mode

Now we will ask, "What are the allowed occupation numbers?" and focus on just one mode of single particle energy ε

Fermions

Fermions have half integer spin and include electrons, protons neutrons

For fermions no two particles can be in the same state, and thus the occupation number for a mode can be either $n = 0$ (unoccupied) or $n = 1$ (occupied)

Then

$$Z_G = \sum_{n=0}^{1} e^{-\beta(n\varepsilon - \mu n)} = \boxed{1 + e^{-\beta(\varepsilon-\mu)} = Z_G}$$

- Now given Z_G we can determine the mean number of particles in the mode

$$\bar{n} = \frac{1}{\beta}\frac{\partial}{\partial\mu}\ln Z_G = \frac{1}{\beta}\frac{\partial}{\partial\mu}\ln\left(1 + e^{-\beta(\varepsilon-\mu)}\right) = \frac{e^{-\beta(\varepsilon-\mu)}}{1 + e^{-\beta(\varepsilon-\mu)}}$$

$$\boxed{\bar{n}_{FD} = \frac{1}{e^{\beta\mu}+1}}$$

- Alternatively, the mean occupation number is

Same as before

$$\bar{n} = \sum_{n=0}^{1} P_n \cdot n = P_1 = \frac{e^{-\beta(\varepsilon-\mu)}}{Z_G} = \frac{e^{-\beta(\varepsilon-\mu)}}{1 + e^{-\beta(\varepsilon-\mu)}}$$

Bosons

- Particles are bosonic if they have spin 0, 1, 2 .... The photon is also a boson, and has two polarization states (right handed and left handed)

- There can be an arbitrary number of quanta (or particles) in a mode for bosonic particles

$$n = 0, 1, 2, 3, \ldots\ldots$$

So

$$Z_G = \sum_{n=0}^{\infty} e^{-\beta(n\varepsilon - n\mu)} = \sum_{n=0}^{\infty} \left(e^{-\beta(\varepsilon-\mu)}\right)^n$$

$$\boxed{Z = \frac{1}{1 - e^{-\beta(\varepsilon-\mu)}}}$$

- And

$$\bar{n} = \frac{1}{\beta}\frac{\partial \ln Z_G}{\partial \mu} = \frac{e^{-\beta(\varepsilon-\mu)}}{1 - e^{-\beta(\varepsilon-\mu)}}$$

$$\boxed{\bar{n}_{BE} = \frac{1}{e^{\beta(\varepsilon-\mu)} - 1}}$$

Summary (For a single mode only)

Superficially Bosons and Fermions seem similar, differing "only" by a sign. As we will see this makes it quite different.

$$\boxed{\bar{n} = \frac{1}{e^{\beta(\varepsilon-\mu)} \mp 1}} \qquad \boxed{\bar{E} = \bar{n}\,\varepsilon}$$

The grand partition fcns and associated potentials seem related too $\Phi_G = -kT \ln Z_G$

$$\boxed{\Phi_G = \pm kT \ln\left(1 \mp e^{-\beta(\varepsilon-\mu)}\right)}$$

Top sign for bosons!