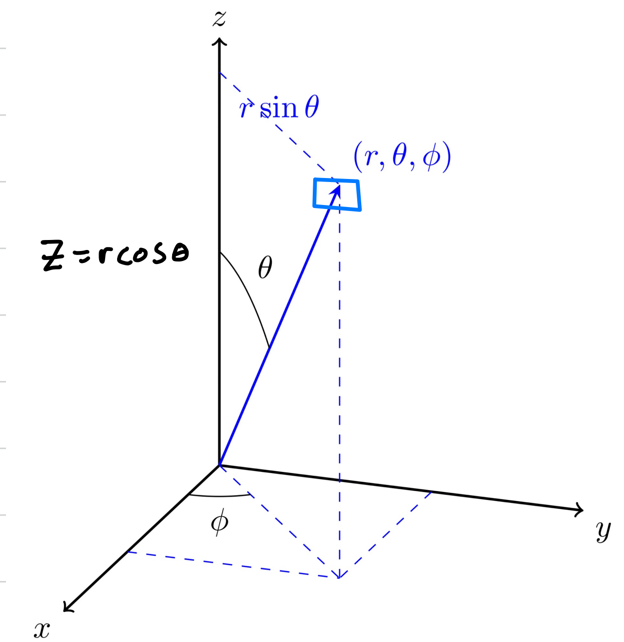
Spherical Coordinates and Solid Angle

We have a particle with position x, y, z. Geometry relates x, y, z to r, θ, φ.



$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

The probability of finding a particle with position in range [x, dx], [y, dy], and [z, dz] is:

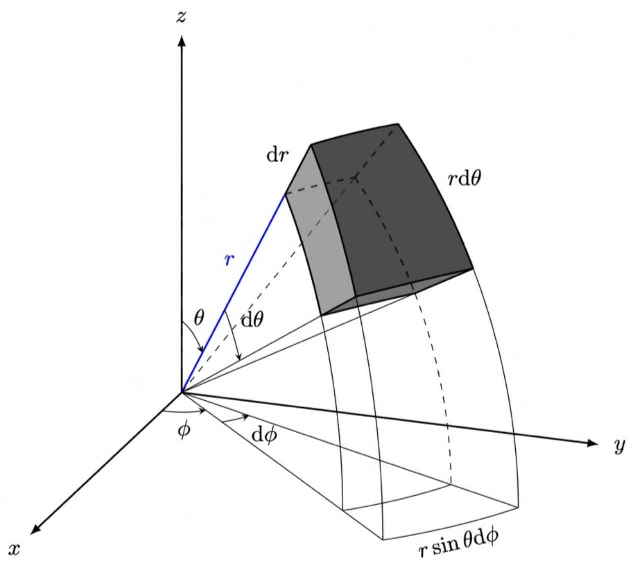
$$d\mathcal{P}_{x,y,z} = P(x,y,z)\ \underbrace{dx\,dy\,dz}_{dV}$$

we need to find this in spherical coords

From the figure

$$\boxed{\begin{aligned} dV &= r^2\,dr\,\sin\theta\,d\theta\,d\phi \\ dA &= r^2\sin\theta\,d\theta\,d\phi \end{aligned}}$$

Spherical Coordinates



Volume and area elements

$$dV = dA\, dr = (r d\theta)\,(r \sin\theta d\phi)\,(dr) = r^2 \sin\theta\, dr d\theta d\phi$$

$$dA = (r d\theta)(r \sin\theta d\phi) = r^2 \sin(\theta)\, d\theta d\phi$$

Then the probability distribution for finding a particles with radius in $[r, dr]$ and angles in $[\theta, d\theta]$, $[\phi, d\phi]$ is

$$d\mathcal{P}_{r,\theta,\phi} = \underbrace{P(x,y,z)\; r^2 \sin\theta}_{\equiv P(r,\theta,\phi)} \; dr\; d\theta\; d\phi$$

The more mathematical way is to work out the Jacobian of the map from $(x,y,z) \mapsto (r,\theta,\phi)$

$$\left| \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} \right| = \begin{vmatrix} \partial x/\partial r & \partial x/\partial\theta & \partial x/\partial\phi \\ \partial y/\partial r & \partial y/\partial\theta & \partial y/\partial\phi \\ \partial z/\partial r & \partial z/\partial\theta & \partial z/\partial\phi \end{vmatrix} = r^2 \sin\theta$$

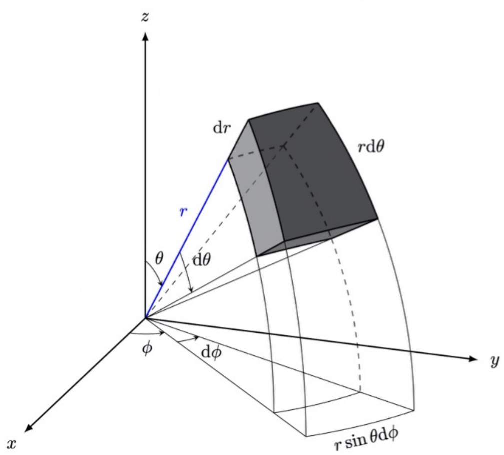
see slide

Then the change of variables formula gives

$$d\mathcal{P}_{r,\theta,\phi} = P(x,y,z) \left\| \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} \right\| dr\, d\theta\, d\phi$$

$$= P(x,y,z)\ r^2 \sin\theta \ \ dr\, d\theta\, d\phi$$

Jacobian Determinant



$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

$$\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\cos\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} = r^2\sin\theta$$

Solid Angle

Consider a sphere. Suppose we have a patch on the sphere of area A. We say that it subtends a "solid angle" of Ω

$$\boxed{\Omega = \frac{A}{r^2}}$$

I don't like the term solid angle: it is not an angle, it is a patch on a sphere parametrized by two angles

(1) The solid angle goes from 0 to 4π. Indeed, $\Omega/4\pi$ is the fraction of the spherical surface covered by the area A.

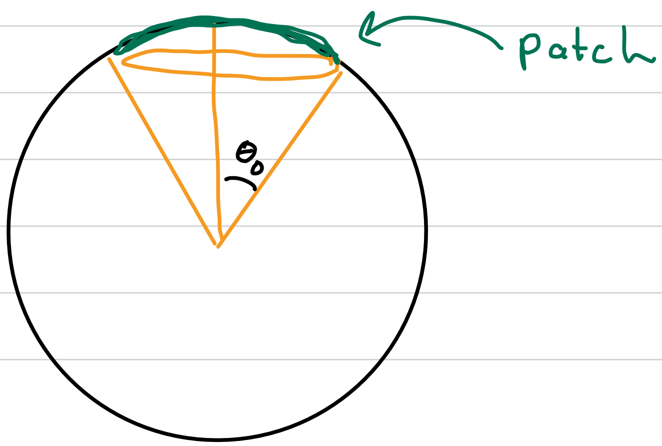
(2) The solid angle generalizes the 1D case where $\theta = s/r$

(3) For a small patch with angular ranges $[\theta, \theta+d\theta]$ and $[\phi, \phi+d\phi]$ We refer back to the picture at the beginning of these notes

$$\boxed{d\Omega = \frac{dA}{r^2} = \frac{r^2 \sin\theta \, d\theta \, d\phi}{r^2} = \sin\theta \, d\theta \, d\phi}$$

called the differential solid angle

(4) The solid angle of a cone is



$$\Omega = \int d\Omega = \int_0^{\theta_0} \sin\theta \, d\theta \int_0^{2\pi} d\phi$$

$$= 2\pi(1 - \cos\theta_0)$$

Example

A particle is equally likely to be anywhere on a sphere (i.e. it is uniformly distributed over the sphere's area). What is the prob dist in terms of θ, ϕ?

$$d\mathcal{P} \propto dA \propto R^2 d\Omega \propto \sin\theta \, d\theta \, d\phi$$

The normalization constant is $1/4\pi$, and so

$$\boxed{d\mathcal{P} = \frac{\sin\theta \, d\theta \, d\phi}{4\pi} = \frac{d\Omega}{4\pi}}$$