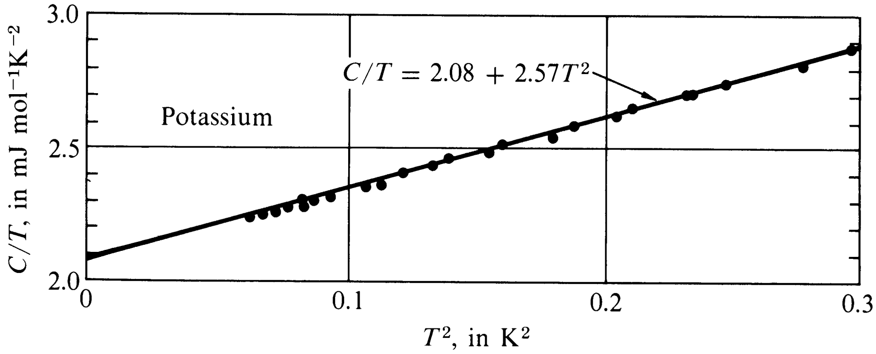

Figure 11: *The low temperature heat capacity of potassium, plotted as* C_V/T *versus* T^2. *From C. Kittel, and H. Kroemer, Themal physics (W.H. Freeman & co., New York NY, 1980).*

Hence,

$$\frac{c_V}{T} = \gamma + A\,T^2. \tag{8.118}$$

If follows that a plot of c_V/T versus T^2 should yield a *straight line* whose intercept on the vertical axis gives the coefficient γ. Figure 11 shows such a plot. The fact that a good straight line is obtained verifies that the temperature dependence of the heat capacity predicted by Eq. (8.117) is indeed correct.

8.14 White-dwarf stars

A main-sequence hydrogen-burning star, such as the Sun, is maintained in equilibrium via the balance of the gravitational attraction tending to make it collapse, and the thermal pressure tending to make it expand. Of course, the thermal energy of the star is generated by nuclear reactions occurring deep inside its core. Eventually, however, the star will run out of burnable fuel, and, therefore, start to collapse, as it radiates away its remaining thermal energy. What is the ultimate fate of such a star?

A burnt-out star is basically a gas of electrons and ions. As the star collapses, its density increases, so the mean separation between its constituent particles decreases. Eventually, the mean separation becomes of order the de Broglie wavelength of the electrons, and the electron gas becomes *degenerate*. Note, that the de Broglie wavelength of the ions is much smaller than that of the electrons, so the ion gas remains non-degenerate. Now, even at zero temperature, a degenerate electron gas exerts a substantial pressure, because the Pauli exclusion principle prevents the mean electron separation from becoming significantly smaller than the typical de Broglie wavelength (see the previous section). Thus, it is possible for a burnt-out star to maintain itself against complete collapse under gravity via the *degeneracy pressure* of its constituent electrons. Such stars are termed *white-dwarfs*. Let us investigate the physics of white-dwarfs in more detail.

The total energy of a white-dwarf star can be written

$$E = K + U, \tag{8.119}$$

where K is the total kinetic energy of the degenerate electrons (the kinetic energy of the ion is negligible) and U is the gravitational potential energy. Let us assume, for the sake of simplicity, that the density of the star is *uniform*. In this case, the gravitational potential energy takes the form

$$U = -\frac{3}{5}\frac{G\,M^2}{R}, \tag{8.120}$$

where G is the gravitational constant, M is the stellar mass, and R is the stellar radius.

Let us assume that the electron gas is highly degenerate, which is equivalent to taking the limit $T \rightarrow 0$. In this case, we know, from the previous section, that the Fermi momentum can be written

$$p_F = \Lambda \left(\frac{N}{V}\right)^{1/3}, \tag{8.121}$$

where

$$\Lambda = (3\pi^2)^{1/3}\,\hbar. \tag{8.122}$$

Here,

$$V = \frac{4\pi}{3}\,R^3 \tag{8.123}$$

is the stellar volume, and N is the total number of electrons contained in the star. Furthermore, the number of electron states contained in an annular radius of **p**-space lying between radii p and $p + dp$ is

$$dN = \frac{3\,V}{\Lambda^3}\,p^2\,dp. \tag{8.124}$$

Hence, the total kinetic energy of the electron gas can be written

$$K = \frac{3\,V}{\Lambda^3}\int_0^{p_F}\frac{p^2}{2\,m}\,p^2\,dp = \frac{3}{5}\,\frac{V}{\Lambda^3}\,\frac{p_F^5}{2\,m}, \tag{8.125}$$

where m is the electron mass. It follows that

$$K = \frac{3}{5}\,N\,\frac{\Lambda^2}{2\,m}\left(\frac{N}{V}\right)^{2/3}. \tag{8.126}$$

The interior of a white-dwarf star is composed of atoms like C^{12} and O^{16} which contain equal numbers of protons, neutrons, and electrons. Thus,

$$M = 2\,N\,m_p, \tag{8.127}$$

where m_p is the proton mass.

Equations (8.119), (8.120), (8.122), (8.123), (8.126), and (8.127) can be combined to give

$$E = \frac{A}{R^2} - \frac{B}{R}, \tag{8.128}$$

where

$$A = \frac{3}{20}\left(\frac{9\pi}{8}\right)^{2/3}\frac{\hbar^2}{m}\left(\frac{M}{m_p}\right)^{5/3}, \tag{8.129}$$

$$B = \frac{3}{5}\,G\,M^2. \tag{8.130}$$

The equilibrium radius of the star R_* is that which *minimizes* the total energy E. In fact, it is easily demonstrated that

$$R_* = \frac{2\,A}{B}, \tag{8.131}$$

which yields

$$R_* = \frac{(9\pi)^{2/3}}{8}\,\frac{\hbar^2}{m}\,\frac{1}{G\,m_p^{5/3}\,M^{1/3}}. \tag{8.132}$$

The above formula can also be written

$$\frac{R_*}{R_\odot} = 0.010\left(\frac{M_\odot}{M}\right)^{1/3}, \tag{8.133}$$

where $R_\odot = 7\times 10^5$ km is the solar radius, and $M_\odot = 2\times 10^{30}$ kg is the solar mass. It follows that the radius of a typical solar mass white-dwarf is about 7000 km: *i.e.*, about the same as the radius of the Earth. The first white-dwarf to be discovered (in 1862) was the companion of Sirius. Nowadays, thousands of white-dwarfs have been observed, all with properties similar to those described above.

8.15 The Chandrasekhar limit

One curious feature of white-dwarf stars is that their radius decreases as their mass increases [see Eq. (8.133)]. It follows, from Eq. (8.126), that the mean energy of the degenerate electrons inside the star increases strongly as the stellar mass increases: in fact, $K \propto M^{4/3}$. Hence, if M becomes sufficiently large the electrons become *relativistic*, and the above analysis needs to be modified. Strictly speaking, the non-relativistic analysis described in the previous section is only valid in the low mass limit $M \ll M_\odot$. Let us, for the sake of simplicity, consider the ultra-relativistic limit in which $p \gg m\,c$.

The total electron energy (including the rest mass energy) can be written

$$K = \frac{3\,V}{\Lambda^3}\int_0^{p_F}(p^2\,c^2 + m^2\,c^4)^{1/2}\,p^2\,dp, \tag{8.134}$$

by analogy with Eq. (8.125). Thus,

$$K \simeq \frac{3\,V\,c}{\Lambda^3}\int_0^{p_F}\left(p^3 + \frac{m^2\,c^2}{2}\,p + \cdots\right)dp, \tag{8.135}$$

giving

$$K \simeq \frac{3}{4}\,\frac{V\,c}{\Lambda^3}\left[p_F^4 + m^2\,c^2\,p_F^2 + \cdots\right]. \tag{8.136}$$

It follows, from the above, that the total energy of an ultra-relativistic white-dwarf star can be written in the form

$$E \simeq \frac{A - B}{R} + C\,R, \tag{8.137}$$

where

$$A = \frac{3}{8}\left(\frac{9\pi}{8}\right)^{1/3}\hbar\,c\left(\frac{M}{m_p}\right)^{4/3}, \tag{8.138}$$

$$B = \frac{3}{5}\,G\,M^2, \tag{8.139}$$

$$C = \frac{3}{4}\,\frac{1}{(9\pi)^{1/3}}\,\frac{m^2\,c^3}{\hbar}\left(\frac{M}{m_p}\right)^{2/3}. \tag{8.140}$$

As before, the equilibrium radius R_* is that which minimizes the total energy E. However, in the ultra-relativistic case, a non-zero value of R_* only exists for $A - B > 0$. When $A - B < 0$ the energy decreases monotonically with decreasing stellar radius: in other words, the degeneracy pressure of the electrons is incapable of halting the collapse of the star under gravity. The criterion which must be satisfied for a relativistic white-dwarf star to be maintained against gravity is that

$$\frac{A}{B} > 1. \tag{8.141}$$

This criterion can be re-written

$$M < M_C, \tag{8.142}$$

where

$$M_C = \frac{15}{64}\,(5\pi)^{1/2}\,\frac{(\hbar\,c/G)^{1/2}}{m_p^2} = 1.72\,M_\odot \tag{8.143}$$

is known as the *Chandrasekhar limit*, after A. Chandrasekhar who first derived it in 1931. A more realistic calculation, which does not assume constant density, yields

$$M_C = 1.4\,M_\odot. \tag{8.144}$$

Thus, if the stellar mass exceeds the Chandrasekhar limit then the star in question cannot become a white-dwarf when its nuclear fuel is exhausted, but, instead, must continue to collapse. What is the ultimate fate of such a star?

8.16 Neutron stars

At stellar densities which greatly exceed white-dwarf densities, the extreme pressures cause electrons to combine with protons to form neutrons. Thus, any star which collapses to such an extent that its radius becomes significantly less than that characteristic of a white-dwarf is effectively transformed into a gas of neutrons. Eventually, the mean separation between the neutrons becomes comparable with their de Broglie wavelength. At this point, it is possible for the degeneracy pressure of the neutrons to halt the collapse of the star. A star which is maintained against gravity in this manner is called a *neutron star*.

Neutrons stars can be analyzed in a very similar manner to white-dwarf stars. In fact, the previous analysis can be simply modified by letting $m_p \to m_p/2$ and $m \to m_p$. Thus, we conclude that non-relativistic neutrons stars satisfy the mass-radius law:

$$\frac{R_*}{R_\odot} = 0.000011 \left(\frac{M_\odot}{M}\right)^{1/3}, \tag{8.145}$$

It follows that the radius of a typical solar mass neutron star is a mere 10 km. In 1967 Antony Hewish and Jocelyn Bell discovered a class of compact radio sources, called *pulsars*, which emit extremely regular pulses of radio waves. Pulsars have subsequently been identified as rotating neutron stars. To date, many hundreds of these objects have been observed.

When relativistic effects are taken into account, it is found that there is a critical mass above which a neutron star cannot be maintained against gravity. According to our analysis, this critical mass, which is known as the *Oppenheimer-Volkoff limit*, is given by

$$M_{OV} = 4\, M_C = 6.9\, M_\odot. \tag{8.146}$$

A more realistic calculation, which does not assume constant density, does not treat the neutrons as point particles, and takes general relativity into account, gives a somewhat lower value of

$$M_{OV} = 1.5\text{—}2.5\, M_{\odot}. \tag{8.147}$$

A star whose mass exceeds the Oppenheimer-Volkoff limit cannot be maintained against gravity by degeneracy pressure, and must ultimately collapse to form a *black-hole*.