

Problem 1. Central Limit Theorem and Random Walk

In a random walk, a collegiate drunkard starts at the origin and takes a step of size a , to the right with probability p and to the left with probability $1 - p$.

- Take $p = 1/2$, i.e. equal probability of right and left steps. Determine the probability of the drunkard having position X , i.e. $P(X)$, after three steps. Plot $P(X)$ where X can be one of $X/a = 0, \pm 1, \pm 2, \pm 3$. Note how your graph begins to approach a Gaussian after just three steps¹
- Now keep p general. What is the mean and variance variance in the drunkard's position X after one step, and after two steps? You can check your reasoning by doing the next part.
- After n steps (with $n \gg 1$) find his mean position $\langle X \rangle$, and the std. deviation in his position $\sigma_X = \sqrt{\langle \delta X^2 \rangle}$. Check your result by comparing with the figure below

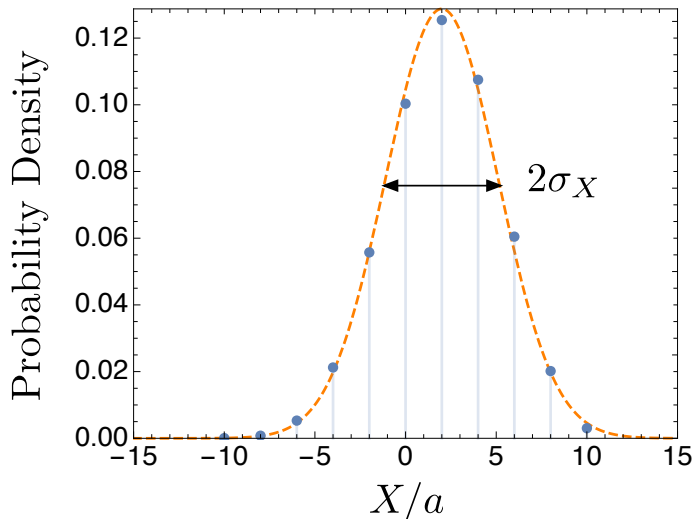


Figure 1: Probability of our drunkard having position X after $n = 10$ steps (the blue points). Of course after 10 steps the drunkard will be between $-10 \dots 10$, and it is easy to show that he will be only at the even sites, i.e. $-10, -8, -6, \dots 10$. For $p = 0.6$, I find $\langle X \rangle = 2.0$. Twice the std deviation, $2\sigma_X$, is shown in the figure and is about six in this case. The orange curve is a gaussian (a.k.a the “bell-shaped” curve) approximation discussed in class and approximately agrees with the points – this is the central limit theorem. Recall that the central limit theorem says that if the number of steps n is large, the probability of X (a sum of n independent events) is approximately $P(x) dX \propto \exp(-(X - \langle X \rangle)^2 / 2\sigma_X^2)$. Evidently the gaussian approximation works well already for $n = 10$.

Hint: X is a sum N independent events x_i where $x_i = \pm a$. Use results from class on the probability distribution of a *sum* of independent events.

¹The graph should be symmetric. You should find $P_0 = 0$, $P_{\pm 1} = \frac{3}{8}$, $P_{\pm 2} = 0$, $P_{\pm 3} = \frac{1}{8}$. Your graph should look something like the figure below but symmetric around the origin.

Problem 2. Big numbers and the Shannon entropy

Here $N_A = 6.0 \times 10^{23}$

- (a) Consider a number

$$e^S = 100000e^{N_A} \quad (1)$$

and an approximation to it

$$e^S = 100000 e^{N_A} \simeq e^{N_A}. \quad (2)$$

What is the percent error in S made by this approximation?

- (b) Consider the approximation

$$e^S = e^{N_A} + e^{1.001N_A} \simeq e^{1.001N_A}. \quad (3)$$

What is the percent error in S made by this approximation? It may be helpful to recall the Taylor series of the logarithm discussed in previous homework.

- (c) Suppose that I have a subsystem which can be in three states, $s = 1, 2, 3$, with probabilities p_s . If I lay down N subsystems drawn from the probability distribution p_s (see Fig. 2 for two concrete examples), then for N large I will have approximately $N_1 \simeq Np_1$ subsystems in state 1, $N_2 \simeq Np_2$ subsystems in state two, and $N_3 \simeq Np_3$ subsystems in state three.

The total number of configurations with specified N_1 , N_2 and N_3 that can be generated during this process is Ω . We study its logarithm $S \equiv \ln \Omega$. We showed in class and in prior homework that

$$S \equiv \ln \Omega = \ln \left(\frac{N!}{N_1!N_2!N_3!} \right) \simeq NS_1 \quad S_1 \equiv \sum_s -p_s \ln p_s \quad (4)$$

where S_1 is known as the Shannon entropy. Thus Ω grows exponentially in the number of subsystems

$$\Omega \simeq e^{NS_1} \quad (5)$$

This formula generalizes straightforwardly to subsystems with more than three states.

- (i) What is the Shannon entropy for the probability distribution leading to Fig. 2(b)? How would this change if my subsystem had two states that are equally likely, or six states that are equally likely?
- (ii) Compute the Shannon entropy for the probability distribution leading to Fig. 2(a)². By comparing Fig. 2(a) and Fig. 2(b) and with no more than a sentence or two, try to qualitatively explain why your result in (ii) is smaller than the case of equal probability discussed in (i).

²Answer: 0.9

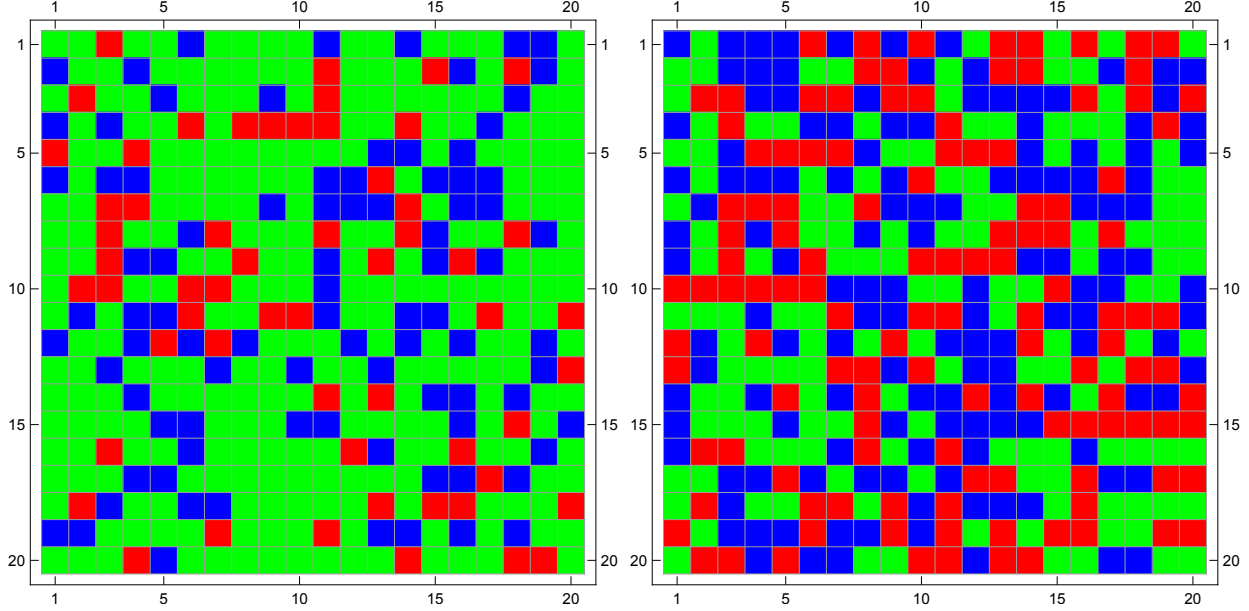


Figure 2: (a) A configuration generated by laying down 400 subsystems, with probabilities $p_1 = \frac{1}{8}$ (red), $p_2 = \frac{2}{8}$ (blue) and $p_3 = \frac{5}{8}$ (green). $\ln \Omega$ is the number of ways you can shuffle around the red, blue, and green and get a new configuration. (b) A configuration generated by laying down 400 subsystems, with probabilities $p_1 = \frac{1}{3}$ (red), $p_2 = \frac{1}{3}$ (blue) and $p_3 = \frac{1}{3}$ (green).

- (iii) Suppose that the subsystem actually describes two independent subsystems A and B , $p_s \equiv p_s^{AB}$. For instance, system A can be in states $a = 1, 2$ with probability p_a^A , and system B can be in states $b = 1, 2, 3$ with probability p_b^B . The probability to be in a state labeled by a and b is

$$p_s^{AB} = p_a^A p_b^B \quad (6)$$

where the six possible states labeled by $s \equiv (a, b)$ are

$$s \equiv (a, b) \in \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3)\} \quad (7)$$

Show that entropy of the total subsystem is the sum of the entropy of A and B

$$S_1^{AB} = S_1^A + S_1^B \quad (8)$$

where

$$S_1^A = \sum_a -p_a^A \ln p_a^A \quad \text{and} \quad S_1^B = \sum_b -p_b^B \ln p_b^B \quad (9)$$

Problem 3. Counting

Consider 400 atoms laid out in a row. Each atom can be in one of two states a ground state with energy 0 and an excited state with energy Δ . Assume that 100 of the atoms are excited, so the total energy is $U = 100 \Delta$.

- (a) Show that there are e^{225} configurations, called microstates, for this energy U . One microstate is shown below.
- (b) Suppose that we make a partition of the energy so that the first 200 atoms have an energy of 80Δ , and the next 200 atoms have an energy of 20Δ (see below). The terminology here is that we have specified the “macrostate” (i.e. the 80/20 split), leaving the microstates (exactly which atoms are up and down) to be further specified. How many microstates are there with this macrostate? One microstate for this 80/20 split macrostate is shown below³

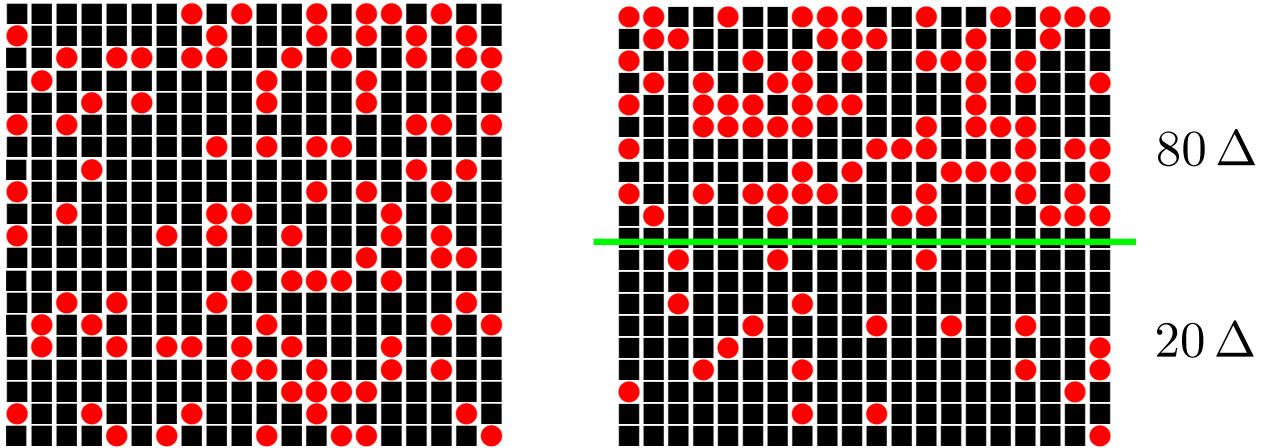


Figure 3: (a) A microstate where the energy is not partitioned. (b) a microstate where the energy is partitioned – 80% on the top and 20% on the bottom.

³Answer: e^{200} .

Problem 4. The Gamma function

The $\Gamma(x)$ function can be defined as⁴

$$\Gamma(x) \equiv \int_0^\infty du e^{-u} u^{x-1} = \int_0^\infty \frac{du}{u} e^{-u} u^x \quad (10)$$

A plot of $\Gamma(x)$ is shown below. $\Gamma(n)$ provides a unique generalization of $(n-1)!$ when n is not an integer and even negative or complex. It will come up a number of times in this course and is good to know.

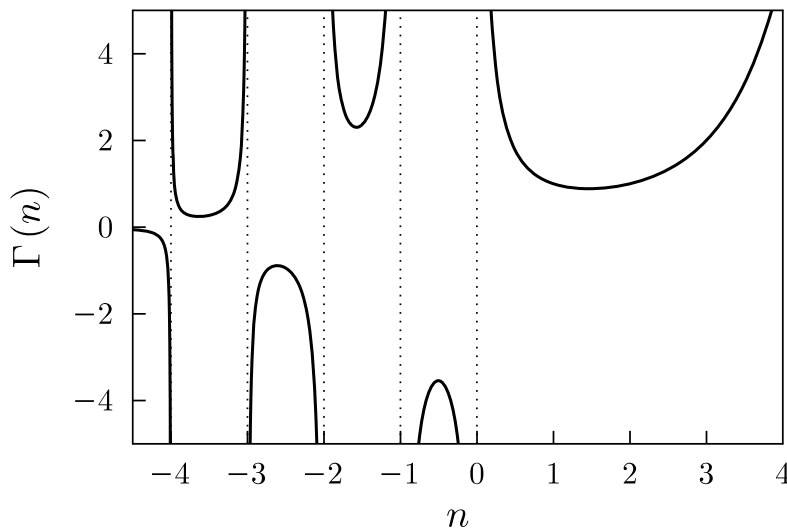


Fig. C.1 The gamma function $\Gamma(n)$ showing the singularities for integer values of $n \leq 0$. For positive, integer n , $\Gamma(n) = (n-1)!$.

Figure 4: Appendix C.2 of our book

- Using notions of generating functions, briefly explain why $\Gamma(n) = (n-1)!$ for n integer.
- Prove that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. *Hint:* try a substitution $y = \sqrt{u}$.

The following identity is needed below.

$$\Gamma(x+1) = x\Gamma(x), \quad (11)$$

or

$$x! = x \cdot (x-1)!, \quad (12)$$

but now x is a real number, and $x!$ is defined by $\Gamma(x+1)$.

- (Optional. Dont turn in) Use integration by parts to prove the identity in Eq. (11).

⁴I like to write $\Gamma(x) = \int_0^\infty \frac{du}{u} e^{-u} u^x$, which makes the x is more explicit. Also the measure du/u is invariant under a homogeneous rescaling, e.g. under change of variables $u \rightarrow u' = \lambda u$ we have $du'/u' = du/u$.

- (d) Use the results of this problem to show that $\Gamma(\frac{7}{2}) = 15\sqrt{\pi}/8$. What is the result numerically? $7/2$ is between two integers. Show that $\Gamma(7/2)$ is between the appropriate factorials related to those two integers?
- (e) The “area” (i.e. circumference) of a “sphere” in two dimensions (i.e. the circle) is $2\pi r$. The area of a sphere in three dimensions is $4\pi r^2$. A general formula for the area of the sphere in d dimensions is derived in the book is (the proof is simple, using what we know)

$$A_d(r) = \frac{2\pi^{d/2}}{(\frac{d}{2} - 1)!} r^{d-1} = \frac{2\pi^{d/2}}{\Gamma(d/2)} r^{d-1} \quad (13)$$

Show that this formula gives the familiar result for $d = 2$ and $d = 3$.

Problem 5. Two State System

Consider an array of N atoms forming a medium, with each atom possessing two energy states: a ground state with energy 0 and an excited state with energy Δ .

- Determine the temperature at which the number of excited atoms reaches $N/4$.
- (Optional) Calculate the mean energy $\langle \epsilon \rangle$. Sketch $\langle \epsilon \rangle / kT$ as a function of Δ / kT . Explain both the high and low temperature limits qualitatively.
- (Optional) Show that the variance of energy $\langle (\delta \epsilon)^2 \rangle$ for an individual atom takes the form

$$\langle (\delta \epsilon)^2 \rangle = \frac{\Delta^2 e^{-\beta \Delta}}{(1 + e^{-\beta \Delta})^2}$$

Graph $\frac{\langle (\delta \epsilon)^2 \rangle}{(kT)^2}$ as a function of $\frac{\Delta}{kT}$.

- (Optional) Suppose you have a collection of 16 such atoms (shown below). Calculate the average values of $\langle E \rangle$, $\langle (\delta E)^2 \rangle$ and $\langle E^2 \rangle$, where E represents the total energy of all 16 atoms. What approximately is the probability distribution for the energy E ?
Hint: Think about how the central limit theorem applies here.

